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J. Phys. A: Math. Gen. **22** (1989) 585-589. Printed in the UK

COMMENT

On Lyapunov and dimension spectra of 2D attractors, with an application to the Lozi map

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Received 25 August 1988

Abstract. We show that the Lyapunov spectra (and thus also the entropy and dimension spectra) of 2D attractors can be obtained from the stretching of any typical line. This is particularly simple for piecewise linear maps. In particular, it gives a very easy way of estimating these spectra for the Lozi map. Comparing with some previously published spectra, we find these to have large errors.

Measures concentrated on fractal sets, such as invariant measures on strange attractors or electrostatic charge distributions on fractal clusters, are typically characterised by a whole spectrum of dimension-like quantities. These spectra and their relation, in the case of dynamical systems, to the spectrum of Lyapunov exponents, has been the subject of numerous investigations in the recent literature [1-17].

In particular, it was conjectured in [7], and proven for hyperbolic systems in [14], that the generalised dimensions $D(q)$ introduced in [1] are related to the Hausdorff dimensions $f(\alpha)$ of sets with pointwise dimension α via a Legendre transform. More precisely, let us define

$$\tau(q)=(q-1)D(q) \tag{1}$$

so then

$$f(\alpha)=\alpha q-\tau(q) \qquad \alpha=\mathrm{d}\tau/\mathrm{d}q \qquad q=\mathrm{d}f/\mathrm{d}\alpha. \tag{2}$$

The pointwise dimension $\alpha(x)$ is defined via the scaling of the masses of ε-balls centred at x, $\mu(x)\sim\varepsilon^{\alpha(x)}$, while $D(q)$ is defined via the $(q-1)$th moment of μ_ε:

$$\langle\mu_\varepsilon^{q-1}\rangle\sim\varepsilon^{\tau(q)}. \tag{3}$$

(The averaging here is done with respect to the measure μ, not with respect to the Lebesgue measure.)

While the dimensions are very easy to compute for strange attractors at the onset of chaos and for systems with simple symbolic dynamics (e.g. Julia sets [9, 10, 18]), their estimation is much less easy for other systems. Even for the quadratic map $x'=a-x^2$ with arbitrary parameter a, or for simple 2D maps such as the Henon map, computing $D(q)$ or $f(\alpha)$ precisely is non-trivial. In these cases, the situation is particularly complicated due to the non-hyperbolicity of the maps, which leads to a phase-transition-like phenomenon [15].

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Rather than by a direct evaluation of (3), the estimate of $f(\alpha)$ and $D(q)$ is easiest via Lyapunov exponents. In the above systems, the Lyapunov exponents are not the same at every point, though they are of course the same nearly everywhere (provided the system is ergodic). Thus, one has fluctuations in the 'effective' Lyapunov exponents $\lambda^{(n)}$ measured over a finite number, n, of iterations [2-6] (in the following, λ will always denote the positive Lyapunov exponent). The $D(q)$ are either obtained perturbatively in $q-1$ from cumulants of $\exp(n\lambda^{(n)})$ [2], from the generating function of these cumulants [5, 6] or from the Legendre transform [11] of this generating function [8, 13]. In addition to the generalised and pointwise dimensions, one can also introduce generalised dynamical entropies $K(q)$, pointwise entropies α_0 and the entropies $f_0(\alpha_0)$ of the set of trajectories with α_0, thus generalising the metric and topological entropies [2, 11]. For natural measures on 2D hyperbolic attractors, the function $\tau_0(q)=(q-1)K(q)$ is essentially the generating function mentioned above [5, 6, 12], α_0 is just the Lyapunov exponent λ and $f_0(\alpha_0)$ is very simply related to $f(\alpha)$. The latter relation becomes particularly simple for maps with constant Jacobian J. With $B=\log|J|$, we obtain [12, 13]:

$$f(\alpha)=1+f_0(\alpha_0)/(\alpha_0+B) \qquad (4)$$

with

$$\alpha=1+\alpha_0/(\alpha_0+B). \qquad (5)$$

These equations seem to provide the easiest way to estimate $f(\alpha)$ and $D(q)$ for 2D hyperbolic attractors, provided the map is given in analytic form.

Up to now, the trajectories from which $f_0(\alpha_0)$ are to be estimated are arbitrary. We can either take randomly chosen chaotic trajectories, or else periodic trajectories. In the latter case, we use the fact that the natural measure can be approximated by a measure concentrated on all periodic orbits with period n, with each orbit carrying a mass [19-21]

$$\mu_{\mathrm{orbit}}=\exp(-\lambda^{(n)}_{\mathrm{orbit}}). \qquad (6)$$

It was claimed in [16, 17, 22] that using periodic orbits is superior not only for some mathematical questions (something well known [23]), but also numerically.

In this comment, I shall present a new way of estimating the Lyapunov spectrum (or, equivalently, the generalised entropies $K(q)$). This method is most easily applied to piecewise linear maps such as the Lozi map [24]. Our numerical results, obtained with rather modest effort, and presented in figures 1 and 2, show that both attempts of [17], to measure $f(\alpha)$ directly from (3) and to measure it from (4) via periodic orbits, contain large errors. In contrast to this, we found perfect numerical agreement with the results of [13].

The new method of estimating $K(q)$ for 2D hyperbolic attractors uses the way that an arbitrary line of finite length L_0 in the basin of attraction is stretched during n iterations, for large n. It is shown in [25] that the total length L_n increases according to the topological entropy, $L_n \sim L_0\exp(nK(0))$. This is easily understood heuristically. Denote by $\mathrm{d}s_n$ the length element of L_n. Then

$$L_n=\int \mathrm{d}s_0\left|\frac{\mathrm{d}s_n}{\mathrm{d}s_0}\right| \underset{n\to\infty}{\sim} \int \mathrm{d}s_0 \exp(n\lambda^{(n)}(s))$$
$$\sim L_0\langle\exp(n\lambda^{(n)})\rangle \sim L_0\exp(nK(0)). \qquad (7)$$

Here, the second equality comes from the fact that, after many iterations, the curve will follow essentially the unstable manifold, for nearly all curves. Similarly, if the curve was chosen typically, the integration over s_0 will essentially be equivalent to integrating over the start point x_0 with Lebesgue measure, which gives the last equation.

The generalisation of (7) is now straightforward. The generalised entropy $K(q)$ is given by

$$\exp[n(1-q)K(q)] \sim \langle \exp[(1-q)n\lambda^{(n)}]\rangle \sim \int \mathrm{d}s_n \left|\frac{\mathrm{d}s_n}{\mathrm{d}s_0}\right|^{-q} \underset{\mathrm{def}}{=} L_n^{(q)}. \quad (8)$$

This is our main result. Practically, we use it as follows: we start with a straight line of length L_0, and iterate n times. We evaluate the last integral by approximating each iterate by a chain of straight pieces. For each piece, it is easy to compute the length and the derivative. The entire program is particularly easy to program in any computer language allowing recursive function calls, as the length can then be computed recursively.

This method is most straightforwardly applied to piecewise linear maps such as the Lozi map [24] $(x, y) \to (1-a|x|+\mathrm{by}, x)$. There, the integral in (8) reduces simply to a finite sum for any finite n, and can be evaluated exactly. The resulting $f(\alpha)$ for the parameters values $a = 1.7$ and $b = 0.5$ used in [17], obtained with $n = 27$, is given in figure 1. This took about 2 h CPU time on an Atari home computer, and required a negligible amount of memeory. We first used (8) to obtain $K(q) = \frac{1}{2}\lim_{n\to\infty}\log(L_n^{(q)}/L_{n-2}^{(q)})$ (since the ratios $L_n^{(q)}/L_{n-1}^{(q)}$ showed large oscillations for $q \geq 2$ with period 2 in n). We then performed the Legendre transform to $f_0(\alpha_0)$, and finally applied (4). As judged from the convergence with n, the errors in figure 1 should be less than the thickness of the line.

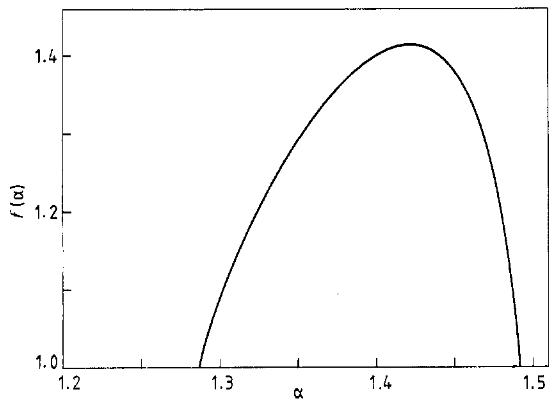

Figure 1. $f(\alpha)$ spectrum for the Lozi map, as estimated from following the expansion of the line $\{-0.5 \leq x \leq 0.5,\ y = 0\}$ during 27 iterations.

In order to compare with other approaches we show this result in figure 2, together with other estimates, and together with the diagonal line $f(\alpha) = \alpha$, to which the curve must be tangential for consistency. The broken curve denoted by a short dash is the result obtained in [17] by evaluating (3) using correlations of points in a random trajectory. Although correct near $q = \mathrm{d}f/\mathrm{d}\alpha = 1$, this is completely wrong for large α (as was to be expected, unless extremely high statistics were used). The light curve is

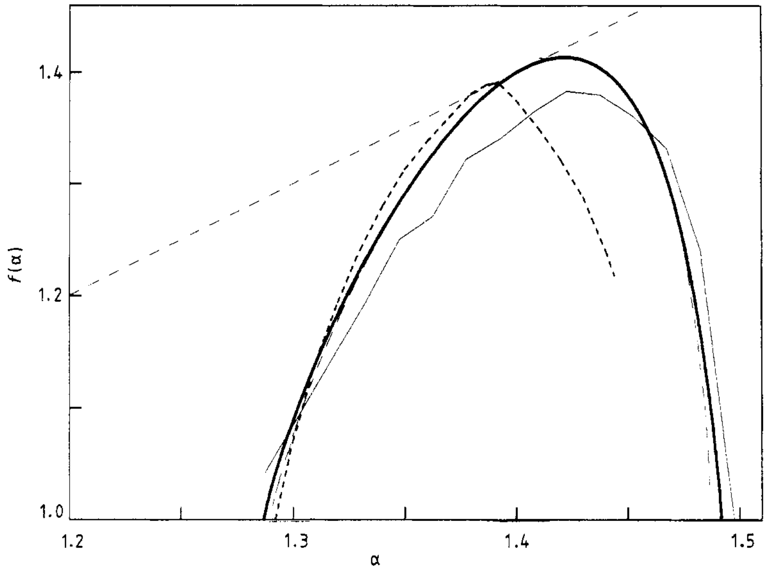

Figure 2. The result of figure 1 (heavy curve) compared to (i) results from a correlation analysis [17] (- - -); (ii) an estimate based on the histogram of the Lyapunov spectrum of all period-17 orbits [17] (light continuous curve); and (iii) an estimate based on the Lyapunov spectrum of a random orbit of length $\sim 10^7$, and using the Legendre transform of the moments $\langle \exp[(1-q)\lambda^{(n)}]\rangle$, with $n = 22$ (— — —).

also from [17], but using all periodic orbits with period $n = 17$, and estimating the distribution of Lyapunov exponents from the histogram. Here the normalisation is completely wrong, which might be due to a wrong application of (4). But apart from this, the lack of smoothness of the curve reflects the fact that there are too few ($\sim 10^3$) periodic orbits of length 17 for making a good histogram. The last entry in figure 2 is the broken curve denoted by a long dash. It is simply obtained by iterating a randomly chosen point approximately 1.5×10^7 times (this took approximately 7h on an Atari), estimating from this the moments of the Lyapunov spectrum (with $n = 22$), Legendre transforming them and finally using (4). Agreement with the 'exact' bold curve is so good that the two curves coincide, except in the wings.

We also studied the parameter values $a = (1+\sqrt{5})/2$ and $b = -0.3$ used in [13]. The differences in $K(q)$ between our results and those of [13] are of the same order of magnitude ($\sim 2 \times 10^{-3}$) as those found within [13] when using periodic and random orbits (neither of which seems to be consistently more precise).

We also tried to apply the method to the Henon map. First of all, it is obvious that (8) develops a singularity at $q = 2$ due to the homoclinic tangencies, in agreement with the arguments of [15]. Thus, we can use (8) only for $q < 2$. Numerically, we approximated the iterates of an initial straight line near the unstable fixed point with $\Delta s_0 = 0.001$ by polygons with $\Delta s_n < 0.001$, for n up to 30. For $1 < q < 2$, the results were not better than those obtained with other methods [2, 12, 13, 15]. But for $q \leqslant 0$, the resulting $K(q)$ seem to be more precise than previous estimates. In particular, we obtain $K(-1) = 0.4908$ and $K(0) = 0.4630$.

In summary, we have given a new method of estimating the $f(\alpha)$ spectrum for 2D maps. Although in principle having a wider application, it is most efficient for piecewise linear maps such as the Lozi map. There it shows that some previous estimates [17] have large errors, but that when applied correctly [13], periodic and chaotic trajectories seem to give comparably precise results.

For other maps with a not-piecewise linear attractor, the method is less efficient, at least in the version used in the present paper. In a more sophisticated approximation one could use splines instead of polygons, but this has not been tried yet.

The method can also be applied to higher-dimensional maps provided they have only one positive Lyapunov exponent. It can also in principle be extended to the case with several (say k) positive λ, but this would require [25] evaluating the expansion of k-dimensional volumes, which seems rather awkward.

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